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ON SOME OPERATORS IN c_0

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ABSTRACT

THEOREM. *Let S be a bounded Suslin set in the plane. Then there is a bounded linear operator T in c_0, whose point spectrum $\sigma_e(T) = S$.*

A similar theorem was obtained in [3], with an operator in a separable space depending on S, and then in [4] in a separable space E independent of S, but far from classical, even though the operator T is of an elementary kind. In c_0 we use a similar operator in a certain subspace of c_0 depending on S, then extend the operator to c_0 by Sobczyk's theorem ([6, 7]) and then adjust the point spectrum of the extension by a further extension to $c_0 + c_0$. An essential role is played by the 3-dimensional analogue of Wiener's theorem on Fourier–Stieltjes coefficients (1924 [2, p. 42]), stated later as Lemma 1.

1. A space of distributions

Let c_{01} be the B-space of squences $x = (x_0, x_1, \ldots, x_n, \ldots)$ such that $\Sigma_0^m |x_n| = o(m)$, with norm $\|x\| = \sup(m+1)^{-1}\Sigma_0^m |x_n|$. Then c_{01} is isomorphic to a subspace of c_0, because each space l_n^1 can be embedded in c_0 by an operator A_n with

$$\|A_n x\| \leqq \|x\| \leqq 2\|A_n x\| \qquad (x \in l_n^1).$$

Let $M_c(T^3)$ be the space of continuous measures μ in T^3 and let $(N_j)_0^\infty$ be an enumeration of the 3-tuples $(n_1, n_2, n_3) \in Z \oplus Z \oplus Z$ (the dual group of T^3), with $\|N_0\| \leqq \|N_1\| \leqq \cdots$. Therefore $N_0 = 0$ and $\|N_j\| \cong aj^3$. We map the measure μ into c_{01} by setting $x_j = \hat{\mu}(N_j)$ for $j \geqq 0$, using

LEMMA 1. *Let μ be a continuous measure in T^3. Then*

$$\sum_{-r}^{r} \sum_{-r}^{r} \sum_{-r}^{r} |\hat{\mu}(n_1, n_2, n_3)|^2 = o(r^3), \qquad r \to +\infty.$$

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This is proved by evaluating the sum explicitly, as in [2, p. 42].

Let $H\mu$ be the element of c_{01} just defined; since the summable sequences are dense in c_{01}, $H(M_c(T^3))$ is dense in c_{01}. For each $j=0,1,2,\dots$ we have

$$|\hat{\mu}(N_j)| \leqq (j+1)\|H\mu\| \leqq a(1+\|N_j\|^3)\|H\mu\|$$

so that c_{01} can be identified with a linear space $\mathscr{H}$ of distribution of T^3.

Lemma 2. *Let* $F \in C^5(T^3)$ *and* $\mu \in M_c(T^3)$. *Then* $\|H(F\cdot\mu)\| \leqq a\|F\|_{C^5}\cdot\|H\mu\|$.

Proof. Let $F = \Sigma \hat{F}(N)\chi_N$, a sum of exponentials. Then

$$|(F\cdot\mu)^\wedge(M)| \leqq \sum_N |\hat{\mu}(M-N)||\hat{F}(N)|.$$

Let $r>0$, and let $|(F\cdot\mu)^\wedge|$ be summed over the set $\|M\|<r$. On this set $\|M-N\|<\|N\|+r$, so the sum can be estimated by

$$a\cdot\sum_N(\|N\|+r+1)^3|\hat{F}(N)|\cdot\|H\mu\| \leqq a(1+r)^3\Sigma(1+\|N\|)^3|\hat{F}(N)|\cdot\|H\mu\|$$
$$\leqq a(1+r)^3\cdot\|F\|_{C^5}\cdot\|H\mu\|,$$

by Parseval's formula applied to the derivatives of F. (The degree of smoothness has no significance.)

Henceforth the multiplication by C^5 functions is defined on all of $\mathscr{H}$, and the support of an element μ of $\mathscr{H}$ is denoted $\Sigma(\mu)$ (see [2, pp. 43–44; 1, §5.6]).

Lemma 3. *Let $\mu \neq 0$ in $\mathscr{H}$; the support of μ is a perfect set.*

Proof. If $\Sigma(\mu)$ contains an isolated point t, then a certain product $\lambda = F\cdot\mu$, with $F\in C^5$, would have support exactly $\{t\}$, and an obvious rotation reduces this to $t=0$.

There must be a smooth function f_1 such that $\langle\lambda, f_1\rangle \neq 0$ but $f_1(0)=0$. Otherwise $\langle\lambda, f\rangle \equiv cf(0)$, and $\lambda \notin \mathscr{H}$ unless $c=0$. Clearly $\langle f_1\cdot\lambda, 1\rangle = \langle\lambda, f_1\rangle \neq 0$; continuing, we can find $f_2,\dots,f_6$ such that $(f_1\cdots f_6)\cdot\lambda_1 \neq 0$. But $f_1\cdots f_6$ is a limit, *in the metric of* C^5, of smooth functions in the ideal $J(0)$, by a theorem of Whitney ([8, p. 638]), so that $(f_1\cdots f_6)\cdot\lambda_1 = 0$.

For any closed set $B \subseteq T^3$, let $\mathscr{H}(B)$ be defined by $\Sigma(\mu)\subseteq B$. Let $F\in C^5$ and let T_F be the multiplication by F: $T_F\mathscr{H}(B)\subseteq\mathscr{H}(B)$.

Lemma 4. *The point spectrum of T_F in $\mathscr{H}(B)$ is precisely the set of z such that $F^{-1}(z)\cap B$ is uncountable.*

PROOF. Suppose that z is in the point spectrum, so that $F \cdot \mu = z\mu$ for some $\mu \neq 0$ in $\mathcal{H}(B)$. Then $\Sigma(\mu) \subseteq F^{-1}(z) \cap B$, and $\Sigma(\mu)$ is uncountable by Lemma 3. Conversely, if $F^{-1}(z) \cap B$ is uncountable, it supports a continuous measure μ; then $\mu \in \mathcal{H}(B)$, and $F \cdot \mu = z\mu$ because μ is an ordinary set-function.

Let now B be a closed set in $|t_1| \leqq 1$, $|t_2| \leqq 1$, $0 \leqq t_3 \leqq 1$, and let $F(t) = t_1 + it_2$. Then F can be extended to be a C^∞ function of (t_1, t_2) of period 2π, i.e. an element of $C^\infty(T^3)$. Identifying B with a subset of $R^3/2\pi Z^3$, we can identify the point spectrum σ_e of the operator T_F in $\mathcal{H}(B)$. According to a theorem of Mazurkiewicz and Sierpiński, σ_e is a Suslin set in the square $|x| \leqq 1$, $|y| \leqq 1$, and any Suslin set S in the square is σ_e for a certain closed set B. (The proof in [5] applies to any set S in a Hausdorff space X, which is the image $f(N)$ of the set of irrational numbers by a continuous mapping of N into X; the place of B is taken by a closed subset of $X \times [0,1]$.) If S is merely bounded, but belongs to the square $|x| \leqq c$, $|y| \leqq c$, then $S = \sigma_e(cT)$, with an operator T_F in $\mathcal{H}(B)$.

$\mathcal{H}(B)$ is isomorphic to a subspace of c_0, whence (Sobczyk's theorem [6], [7]) T admits an extension to an operator $T_1 \in L(c_0)$. Now $\sigma_e(T_1)$ appears to be intractable by this method, but this defect is removed in

LEMMA 5. *Let X be a separable Banach space, Y a closed subspace, and $T_1 \in L(X)$. There is an operator $T_2 \in L(X \oplus c_0)$ such that*

(i) *$T_2 = T_1$ in Y,*

(ii) *all characteristic vectors of T_2 belong to Y.*

Lemma 5 depends on

LEMMA 6. *In c_0 there is an operator A and a subspace $E \cong c_0$, such that*

(iii) *$\|A\| \leqq 1$ and A has no point spectrum,*

(iv) *for all λ, with $|\lambda| < 1$*

$$(\lambda I - A)c_0 \cap E = (0).$$

To deduce Lemma 5 from this, choose $a > \|T_1\|$ and let $R \in L(X, E)$ be defined so that $R^{-1}(0) = Y$. (This is trivial because X/Y is separable and E has a basis). Now

$$T_2(x, z) = (T_1 x, Rx + aAz) \qquad (x \in X, z \in c_0).$$

Assuming that $T_2(x, z) = (\lambda x, \lambda z)$, we see at once from (iii) that $x = 0$ implies $z = 0$. When $x \neq 0$, then $T_1 x = \lambda x$ forces $|\lambda| \leqq \|T_1\| < a$.

But $Rx = (\lambda I - aA)z$ whence by (iv), $Rx = 0$, so $z = 0$, $x \in Y$.

PROOF OF LEMMA 5. Let U be the unilateral shift $U(t_1, t_2, t_3, \ldots) =$

$(0, t_1, t_2, t_3, \ldots)$ so that $U - \lambda I$ is 1-1 in the space of all sequences. Then $(U - \lambda I)z = (1, 0, \ldots)$ can be solved only with $\lambda \neq 0$, but z is unbounded if $|\lambda| < 1$. Since $c_0 = c_0 \oplus c_0 \oplus \cdots$, we can choose $A = U \oplus U \oplus \cdots$, with an obvious definition of E.

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